Turbulence Statistics in a Spectral-Element Code: A Toolbox for High-Fidelity Simulations

Mathematics and Computer Science Division

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Turbulence statistics in a spectral-element code: a toolbox for high-fidelity simulations

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1. Introduction

In the present document we describe a toolbox for the spectral-element code Nek5000 [1], aimed at computing turbulence statistics. The toolbox is presented for a small test case, namely a square duct with $L_x = 2h$, $L_y = 2h$ and $L_z = 4h$, where x , y and z are the horizontal, vertical and streamwise directions, respectively. The number of elements in the xy -plane is $16 \times 16 = 256$, and the number of elements in z is 4, leading to a total of 1,204 spectral elements. A polynomial order of $N = 5$ is chosen, and the mesh is generated using the Nek5000 tool `genbox`. The toolbox presented here allows to compute mean-velocity components, the Reynolds-stress tensor as well as turbulent kinetic energy (TKE) and Reynolds-stress budgets. Note that the present toolbox allows to compute turbulence statistics in turbulent flows with one homogeneous direction (where the statistics are based on time-averaging as well as averaging in the homogeneous direction), as well as in fully three-dimensional flows (with no periodic directions, where only time-averaging is considered). The structure of the `StatsToolbox` folder is as follows:

- (i) The folder `compile` includes the files necessary to compile the main code, both when running with 2D statistics (*i.e.*, for flows with one homogeneous direction) and with 3D statistics.
- (ii) The folder `run` contains all the files necessary to run the case with the statistics.
- (iii) The folders `cpost3d` and `post3d` contain the files necessary to compile and run the code used to postprocess the files obtained during the run when using the 3D version of the statistics.
- (iv) The folder `post2d` contains all the necessary files to compile and run the postprocessing code corresponding to the 2D version of the statistics.

The usage of the statistics toolbox is described below.

2. Description of the statistics toolbox

The statistics code is located in the `compile` folder. In particular, the `userchk` subroutine from the main, `duct usr` file, calls the driver of the statistics `stat_avg`. This subroutine is located at `./stats/statistics.f`, which contains the main code for both the 2D and 3D statistics. All the variables used in `statistics.f` are defined in the common block `./inc_src/STATS`, which contains a very important parameter: `STATS3D`. If this parameter is set to 1, then the 3D version of the statistics is used, whereas if this parameter is set to 0 the 2D version will be used. Note that this parameter is important since it defines the sizes of the variables when compiling, *i.e.*, in the 2D version of the statistics the variables are two-dimensional since the averaging in the homogeneous direction is also performed while running the simulation.

Four run-time parameters are required by the statistics toolbox, and they are all set in the `.rea` file located at `./run/duct.rea`. The parameters under consideration are the following

- (i) `p068 STAT_COMP`. This parameter determines the number of time steps between individual samples used to obtain the time-averaged statistics. The default value is `STAT_COMP=10`, and further insight on the recommended values of `STAT_COMP` to obtain converged statistics is given in Ref. [2].

- (ii) p069 STAT_OUTP. This parameter determines the number of time steps before writing statistic files on disk. These files will be described in detail below, but a general rule is to write two statistic files per simulation. Thus, if p011 NSTEPS is 1000, then p069 STAT_OUTP would be set to 500.
- (iii) p070 CHKPTSTEP. This parameter determines the number of time steps before writing restart files on disk. We will not describe these files in detail in the present document, but the general rule is as with p069 STAT_OUTP, namely to set this parameter so that two sets of restart files are written per simulation.
- (iv) p071 IFCHKPTRST. This parameter determines whether the simulation is started from the initial condition provided in the `useric` subroutine, or it is started from a restart field from a previous simulation. This will not be discussed in detail in the present document, but the idea is that when p071 IFCHKPTRST is set to 1 then the simulation is initiated from the restart fields.

When the subroutine driving the computation of turbulence statistics, *i.e.* `stat_avg`, is called for the first time, all the variables are initialized together with the values of the parameters defined in the `.rea` file. If the current step is a multiple of `STAT_COMP` then the subroutine `stat_compute` is called to perform the statistic computation, and if the step is a multiple of `STAT_OUTP` then the statistics files are written to disk. Note that if the 3D version of the statistics is used then the subroutine `stat_mfo_outfld3D` is used to write the fields, and if the 2D version is used then a communication step is performed first through the subroutine `stat_gl_collation`, and afterwards the writing operation is carried out through the subroutine `stat_mfo_outfld2D`. The writing operations use the `.f` file structure within Nek5000.

Regarding the `stat_compute` subroutine, it is based on the idea of performing time-averaging through the following process:

- 1) The time interval between the last sample taken to compute statistics and the current one is calculated and stored in the variable `dtime`.
- 2) The total time period over which the current statistic file is being averaged is stored in the variable `stat_atime`.
- 3) The present sample represents a fraction of time with respect to the total accumulated statistics of `dtime/stat_atime`, and this fraction is stored in variable `beta`.
- 4) Thus, the previously accumulated statistical quantities represent a fraction of time corresponding to $1 - \text{beta}$, a value that is stored in the variable `alpha`.
- 5) The running time-average of a quantity ϕ , for which an accumulated time-average ϕ_a has been obtained so far and for which a new statistical sample ϕ_n has just been obtained, is computed as: $\bar{\phi} = \alpha\phi_a + \beta\phi_n$. Note that here $\bar{\phi}$ denotes the running time-average of ϕ .

All the terms used to compute the various tensors and budgets, both in the 2D and the 3D versions of the statistics, are described in §4. Note that the process for time-averaging outlined above is the main operation performed in the 3D version of the statistics, whereas in the 2D version the averaging over the homogeneous direction is performed in addition to it. For instance, the following code is used in the `stat_compute` subroutine to compute the average of the horizontal velocity component:

```
! <u>t
if (STATS3D.eq.0) then
  lnvar = lnvar + 1
  npos = lnvar
  call stat_compute_1Dav1(slv1(1,1,1,1,1),npos,alpha,beta)
else
  call add2sxy(STAT(1,npos),alpha,vx,beta,ntot)
endif
```

If the 3D version of the statistics is considered, then the subroutine `add2sxy` from the `math.f` file in the core of Nek5000 is used to compute the time average of the `vx` variable as described above, in

terms of alpha and beta. On the other hand, if the 2D version of the statistics is considered, then the `stat_compute_1Dav1` subroutine is called. In this routine, the averaging in the homogeneous direction is performed first, then the time-averaging is carried out in a similar way as above, using the `add2sxy` subroutine. Note that a similar process is used in more complicated terms, such as for instance in the horizontal velocity fluctuation term. This term would be computed as: $\overline{u^2} = \overline{\tilde{u}\tilde{u}} - U^2$, where $\overline{\tilde{u}\tilde{u}}$ is the time-average of the square of the instantaneous signal $\tilde{u}$, minus the square of the mean value U . Since the mean value is only available after averaging all the statistic files, in the run-time statistics code we compute the term $\overline{\tilde{u}\tilde{u}}$, and the final value $\overline{u^2}$ is computed in a post processing step. The code employed to compute $\overline{\tilde{u}\tilde{u}}$ is the following:

```
! <uu>t
if (STATS3D.eq.0) then
  lnvar = lnvar + 1
  npos = lnvar
  call stat_compute_1Dav2(slv(1,1,1,1,1),slv(1,1,1,1,1),
    $ npos,alpha,beta)
else
  call col3(STAT_UU,vx,vx,ntot)
  call add2sxy(STAT(1,npos),alpha,STAT_UU,beta,ntot)
endif
```

In the 3D version of the statistics, the subroutine `col3` from the `math.f` file is used first to compute the square of `vx`, and then the subroutine `add2sxy` is used to perform the time-average operation as above. A similar process is carried out in the 2D version of the statistics, where the subroutine `stat_compute_1Dav2` is a modified version of `stat_compute_1Dav1` which performs the averaging in the homogeneous direction of the square of `vx` first, then it calculates the time average through `add2sxy`. In the basic version of the toolbox presented here, a total of 44 quantities are computed and stored, both in the 2D and 3D version of the statistics. Note that this is defined by the parameter `STAT_LVAR` in the `STATS` common block. The averaged quantities are stored in the variable `STAT_RUAVG` in the case of the 2D statistics, and in `STAT` in the 3D version. Note however that only one set of variables (2D or 3D) is compiled, as indicated by the parameter `STATS3D`. Moreover, the averaged quantities are stored in the same field number in both versions of the statistics, as summarized in Table 1. A more detailed description of the various terms under consideration is given in §4, and further information regarding turbulence statistics can be found in, for instance, Ref. [3].

3. Postprocessing of the statistics files

3.1. Postprocessing using the 3D version of the statistics

When running the 3D version of the statistics toolbox, the result is the time-average of the 44 variables summarized in Table 1. Since no averaging is performed in any homogeneous direction, the resulting fields are three-dimensional. These are stored in a total of 11 files, named from `s01duct0.f00001` to `s11duct0.f00001`. Note that one typically writes two sets of statistics files per simulation, which means that at the end of the run one would obtain 11 fields with `.f00001` and 11 more with `.f00002`. The time-average of these two would constitute the time-average of the full run. If the complete case under consideration is split into, for instance, 10 runs, then the time-average of each of the 11 files would be obtained from 20 individual fields. The postprocessing code used to perform this operation is located at `./cpost3d/duct.f`. In this folder one sets up the `Nek5000` with the same `SIZE` parameters as the original case, since the postprocessing files are also three-dimensional. In this code, the variable `mtimes` defines the beginning of the time-averaging period.

After setting up the case and compiling, the postprocessing code is run on the directory `./post3d`, where `Nek5000` is set to postprocessing mode by setting the parameter `p011 NSTEPS` from the `.rea` file to 0. The number of fields to read and average in time, *i.e.*, 20 in the example described above, is set through the parameter `p068` from the `.rea` file. This code computes all the statistics described in §4,

Variable	Position	Variable	Position
<u>t	1	<pdwdz>t	23
<v>t	2	<uuu>t	24
<w>t	3	<vvv>t	25
<p>t	4	<www>t	26
<uu>t	5	<uuv>t	27
<vv>t	6	<uuv>t	28
<ww>t	7	<vvu>t	29
<pp>t	8	<vvw>t	30
<uv>t	9	<wwu>t	31
<vw>t	10	<wwv>t	32
<uw>t	11	<ppp>t	33
<pu>t	12	<pppp>t	34
<pv>t	13	<uvw>t	35
<pw>t	14	<uuuu>t	36
<pdudx>t	15	<vvvv>t	37
<pdudy>t	16	<www>t	38
<pdudz>t	17	<e11>t = (du/dx)**2 + (du/dy)**2 + (du/dz)**2	39
<pdvdx>t	18	<e22>t = (dv/dx)**2 + (dv/dy)**2 + (dv/dz)**2	40
<pdvdy>t	19	<e33>t = (dw/dx)**2 + (dw/dy)**2 + (dw/dz)**2	41
<pdvdz>t	20	<e12>t = (du/dx)*(dv/dx) + (du/dy)*(dv/dy) + (du/dz)*(dv/dz)	42
<pdwdx>t	21	<e13>t = (du/dx)*(dw/dx) + (du/dy)*(dw/dy) + (du/dz)*(dw/dz)	43
<pdwdy>t	22	<e23>t = (dv/dx)*(dw/dx) + (dv/dy)*(dw/dy) + (dv/dz)*(dw/dz)	44

Table 1: List of the 44 variables computed in the statistics toolbox, and position where they are stored in the corresponding variable, *i.e.*, STAT_RUAVG in the case of the 2D statistics, and STAT in the 3D version. Note that in the nomenclature employed here, <u>t denotes time-average of the time signal $\tilde{u}$ in the 3D version; in the 2D version of the code, averaging in the homogeneous direction is performed in addition to the time-averaging.

including mean flow, Reynolds-stress tensor and budgets, and outposts all these terms in three-dimensional fields. The result is a total of 20 files, named a01duct0.f00001 to a20duct0.f00001, containing all the three-dimensional turbulence statistics. These fields can then be visualized in VisIt, or further analyzed.

3.2. Postprocessing using the 2D version of the statistics

In the 2D version of the statistics, the result is the average in time and in the homogeneous direction of the 44 variables summarized in Table 1, which implies that the resulting fields are two-dimensional. These are stored in a single file named stduct0.f00001, and as in §3.1 the typical approach is to save two files per simulation: .f00001 and .f00002. The postprocessing code for the 2D version of the statistics is located at ./post2d/duct.f, where Nek5000 is run in postprocessing mode (namely with the parameter p011 NSTEPS from the .rea file set to 0), but also with the dimension in the SIZE file set to 2, given the fact that the statistics files are two-dimensional.

The variables in the stduct0.f files are defined in the 2D version of the spectral-element mesh used to run the simulation. However, it is often convenient to define a different mesh, where the following analysis can be performed more conveniently, for instance without the difficulties introduced by the Gauss-Lobatto-Legendre (GLL) distribution of points within elements. This mesh is defined in the Matlab script mesh_G.m, which produces two binary files with the mesh coordinates: x.fort and y.fort. These are stored in the folder ./ZSTAT. As in §3.1, the number of statistics fields to average is set through the parameter p068 from the .rea file, and the beginning of the time-averaging interval is defined through the variable mtimes. Then, when running this code all the stduct0.f files will be read, averaged in time, and then interpolated to the mesh defined through x.fort and y.fort. The result is stored in the file int_fld, which is also located in the folder ./ZSTAT. Finally, the Matlab script duct_stat.m reads the int_fld, and computes

all the turbulence statistics, including mean flow, Reynolds-stress tensor and the corresponding budgets. The final result is therefore expressed as a series of 2D matrices in Matlab, which can be further analyzed.

4. Description of the terms

In the following, x , y and z denote the three spatial coordinates, whereas $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ are the corresponding instantaneous velocity components. Time-averages are denoted by $(\bar{\cdot})$ or capital letters (U , V and W are the mean velocity components). Lower case corresponds to fluctuating velocities, so the velocity fluctuation is defined as: $\overline{u^2} = \overline{\tilde{u}^2} - U^2$. Note that index summation notation is also used when required. Therefore, the mean velocities constitute a tensor of rank 1:

$$U_i = \begin{pmatrix} U \\ V \\ W \end{pmatrix}, \quad (1)$$

and the (symmetric) Reynolds-stress tensor has rank 2:

$$\overline{u_i u_j} = \begin{pmatrix} \overline{u^2} & \overline{uv} & \overline{uw} \\ \overline{uv} & \overline{v^2} & \overline{vw} \\ \overline{uw} & \overline{vw} & \overline{w^2} \end{pmatrix}. \quad (2)$$

Another second order tensor corresponds to the velocity gradients of the mean velocities:

$$\partial_i U_j = \begin{pmatrix} \partial U/\partial x & \partial U/\partial y & \partial U/\partial z \\ \partial V/\partial x & \partial V/\partial y & \partial V/\partial z \\ \partial W/\partial x & \partial W/\partial y & \partial W/\partial z \end{pmatrix}. \quad (3)$$

The skewness is a third-order tensor related to the triple-product terms, and it is defined as:

$$S_{i,j,1} = \begin{pmatrix} \overline{u^3} & \overline{u^2 v} & \overline{u^2 w} \\ \overline{u^2 v} & \overline{uv^2} & \overline{uvw} \\ \overline{u^2 w} & \overline{uvw} & \overline{uw^2} \end{pmatrix}, \quad S_{i,j,2} = \begin{pmatrix} \overline{u^2 v} & \overline{uv^2} & \overline{uvw} \\ \overline{uv^2} & \overline{v^3} & \overline{v^2 w} \\ \overline{uvw} & \overline{v^2 w} & \overline{vw^2} \end{pmatrix}, \quad S_{i,j,3} = \begin{pmatrix} \overline{u^2 w} & \overline{uvw} & \overline{uw^2} \\ \overline{uvw} & \overline{v^2 w} & \overline{vw^2} \\ \overline{uw^2} & \overline{vw^2} & \overline{w^3} \end{pmatrix}. \quad (4)$$

The transport equation of the Reynolds-stress tensor can be written as:

$$\frac{\partial}{\partial t} \overline{u_i u_j} = P_{ij} + \varepsilon_{ij} + D_{ij} + T_{ij} - \Pi_{ij}^s + \Pi_{ij}^t - C_{ij}, \quad (5)$$

which involves the following tensors of rank 2: P_{ij} is the production, ε_{ij} is the dissipation, D_{ij} is the viscous diffusion, T_{ij} is the turbulent transport, Π_{ij}^s is the pressure strain, Π_{ij}^t is the pressure transport and C_{ij} is the convection. Time is denoted by t . The various tensors are defined as (where ρ and ν denote the fluid density and kinematic viscosity respectively, and p corresponds to the pressure):

$$P_{ij} = -\overline{u_i u_k} \frac{\partial U_j}{\partial x_k} - \overline{u_j u_k} \frac{\partial U_i}{\partial x_k}, \quad (6)$$

$$\varepsilon_{ij} = -2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}}, \quad (7)$$

$$D_{ij} = \nu \frac{\partial^2}{\partial x_k^2} \overline{u_i u_j}, \quad (8)$$

$$T_{ij} = -\frac{\partial}{\partial x_k} \overline{u_i u_j u_k}, \quad (9)$$

$$\Pi_{ij}^s = -\frac{1}{\rho} \left(p \frac{\partial u_i}{\partial x_j} + p \frac{\partial u_j}{\partial x_i} \right), \quad (10)$$

$$\Pi_{ij}^t = -\frac{1}{\rho} \left(\frac{\partial}{\partial x_j} \overline{pu_i} + \frac{\partial}{\partial x_i} \overline{pu_j} \right). \quad (11)$$

$$C_{ij} = U_k \frac{\partial}{\partial x_k} \overline{u_i u_j}. \quad (12)$$

The pressure transport and the pressure strain tensors can be combined to form the velocity-pressure-gradient tensor Π_{ij} , defined as follows:

$$\Pi_{ij} = -\frac{1}{\rho} \left(\overline{u_i \frac{\partial p}{\partial x_j}} + \overline{u_j \frac{\partial p}{\partial x_i}} \right). \quad (13)$$

Using the chain rule:

$$\frac{\partial}{\partial x_i} \overline{pu_j} = \overline{u_j \frac{\partial p}{\partial x_i}} + \overline{p \frac{\partial u_j}{\partial x_i}}, \quad (14)$$

which implies that the velocity-pressure-gradient tensor can be computed as:

$$\Pi_{ij} = \Pi_{ij}^t - \Pi_{ij}^s. \quad (15)$$

It is also possible to write a transport equation for the turbulent kinetic energy (TKE), defined as $k = 1/2 (\overline{u^2} + \overline{v^2} + \overline{w^2})$:

$$\frac{\partial k}{\partial t} = P^k + \varepsilon^k + D^k + T^k - \Pi^{k,s} + \Pi^{k,t} - C^k, \quad (16)$$

where each of the terms can be obtained from the respective tensor, as exemplified here for the case of TKE production:

$$P^k = \frac{1}{2} (P_{xx} + P_{yy} + P_{zz}). \quad (17)$$

4.1. Components of the production tensor

The following terms constitute the various components of the production tensor P_{ij} , both in 2D and in 3D:

$$P_{xx}^{2D} = -2 \left(\overline{u^2} \frac{\partial U}{\partial x} + \overline{uv} \frac{\partial U}{\partial y} \right), \quad (18)$$

$$P_{xx}^{3D} = -2 \left(\overline{u^2} \frac{\partial U}{\partial x} + \overline{uv} \frac{\partial U}{\partial y} + \overline{uw} \frac{\partial U}{\partial z} \right). \quad (19)$$

$$P_{yy}^{2D} = -2 \left(\overline{uv} \frac{\partial V}{\partial x} + \overline{v^2} \frac{\partial V}{\partial y} \right), \quad (20)$$

$$P_{yy}^{3D} = -2 \left(\overline{uv} \frac{\partial V}{\partial x} + \overline{v^2} \frac{\partial V}{\partial y} + \overline{vw} \frac{\partial V}{\partial z} \right). \quad (21)$$

$$P_{zz}^{2D} = -2 \left(\overline{uw} \frac{\partial W}{\partial x} + \overline{vw} \frac{\partial W}{\partial y} \right), \quad (22)$$

$$P_{zz}^{3D} = -2 \left(\overline{uw} \frac{\partial W}{\partial x} + \overline{vw} \frac{\partial W}{\partial y} + \overline{w^2} \frac{\partial W}{\partial z} \right), \quad (23)$$

$$P_{xy}^{2D} = - \left(\overline{u^2} \frac{\partial V}{\partial x} + \overline{uv} \frac{\partial U}{\partial x} + \overline{uv} \frac{\partial V}{\partial y} + \overline{v^2} \frac{\partial U}{\partial y} \right), \quad (24)$$

$$P_{xy}^{3D} = - \left(\overline{u^2} \frac{\partial V}{\partial x} + \overline{uv} \frac{\partial U}{\partial x} + \overline{uv} \frac{\partial V}{\partial y} + \overline{v^2} \frac{\partial U}{\partial y} + \overline{uw} \frac{\partial V}{\partial z} + \overline{vw} \frac{\partial U}{\partial z} \right). \quad (25)$$

$$P_{xz}^{2D} = - \left(\overline{u^2} \frac{\partial W}{\partial x} + \overline{uw} \frac{\partial U}{\partial x} + \overline{uv} \frac{\partial W}{\partial y} + \overline{vw} \frac{\partial U}{\partial y} \right), \quad (26)$$

$$P_{xz}^{3D} = - \left(\overline{u^2} \frac{\partial W}{\partial x} + \overline{uw} \frac{\partial U}{\partial x} + \overline{uv} \frac{\partial W}{\partial y} + \overline{vw} \frac{\partial U}{\partial y} + \overline{uw} \frac{\partial W}{\partial z} + \overline{w^2} \frac{\partial U}{\partial z} \right). \quad (27)$$

$$P_{yz}^{2D} = - \left(\overline{uv} \frac{\partial W}{\partial x} + \overline{uw} \frac{\partial V}{\partial x} + \overline{v^2} \frac{\partial W}{\partial y} + \overline{vw} \frac{\partial V}{\partial y} \right), \quad (28)$$

$$P_{yz}^{3D} = - \left(\overline{uv} \frac{\partial W}{\partial x} + \overline{uw} \frac{\partial V}{\partial x} + \overline{v^2} \frac{\partial W}{\partial y} + \overline{vw} \frac{\partial V}{\partial y} + \overline{vw} \frac{\partial W}{\partial z} + \overline{w^2} \frac{\partial V}{\partial z} \right). \quad (29)$$

4.2. Components of the dissipation tensor

The following terms constitute the various components of the dissipation tensor ε_{ij} , which are the same in 2D and in 3D:

$$\varepsilon_{xx}^{2D,3D} = -2\nu \left[\overline{\left(\frac{\partial u}{\partial x} \right)^2} + \overline{\left(\frac{\partial u}{\partial y} \right)^2} + \overline{\left(\frac{\partial u}{\partial z} \right)^2} \right]. \quad (30)$$

$$\varepsilon_{yy}^{2D,3D} = -2\nu \left[\overline{\left(\frac{\partial v}{\partial x} \right)^2} + \overline{\left(\frac{\partial v}{\partial y} \right)^2} + \overline{\left(\frac{\partial v}{\partial z} \right)^2} \right]. \quad (31)$$

$$\varepsilon_{zz}^{2D,3D} = -2\nu \left[\overline{\left(\frac{\partial w}{\partial x} \right)^2} + \overline{\left(\frac{\partial w}{\partial y} \right)^2} + \overline{\left(\frac{\partial w}{\partial z} \right)^2} \right]. \quad (32)$$

$$\varepsilon_{xy}^{2D,3D} = -2\nu \left[\overline{\frac{\partial u}{\partial x} \frac{\partial v}{\partial x}} + \overline{\frac{\partial u}{\partial y} \frac{\partial v}{\partial y}} + \overline{\frac{\partial u}{\partial z} \frac{\partial v}{\partial z}} \right]. \quad (33)$$

$$\varepsilon_{xz}^{2D,3D} = -2\nu \left[\overline{\frac{\partial u}{\partial x} \frac{\partial w}{\partial x}} + \overline{\frac{\partial u}{\partial y} \frac{\partial w}{\partial y}} + \overline{\frac{\partial u}{\partial z} \frac{\partial w}{\partial z}} \right]. \quad (34)$$

$$\varepsilon_{yz}^{2D,3D} = -2\nu \left[\overline{\frac{\partial v}{\partial x} \frac{\partial w}{\partial x}} + \overline{\frac{\partial v}{\partial y} \frac{\partial w}{\partial y}} + \overline{\frac{\partial v}{\partial z} \frac{\partial w}{\partial z}} \right]. \quad (35)$$

4.3. Components of the viscous diffusion tensor

The following terms constitute the various components of the viscous diffusion tensor D_{ij} , which are the same in 2D and in 3D:

$$D_{xx}^{2D,3D} = \nu \left(\frac{\partial^2}{\partial x^2} \overline{u^2} + \frac{\partial^2}{\partial y^2} \overline{u^2} + \frac{\partial^2}{\partial z^2} \overline{u^2} \right). \quad (36)$$

$$D_{yy}^{2D,3D} = \nu \left(\frac{\partial^2}{\partial x^2} \overline{v^2} + \frac{\partial^2}{\partial y^2} \overline{v^2} + \frac{\partial^2}{\partial z^2} \overline{v^2} \right). \quad (37)$$

$$D_{zz}^{2D,3D} = \nu \left(\frac{\partial^2}{\partial x^2} \overline{w^2} + \frac{\partial^2}{\partial y^2} \overline{w^2} + \frac{\partial^2}{\partial z^2} \overline{w^2} \right). \quad (38)$$

$$D_{xy}^{2D,3D} = \nu \left(\frac{\partial^2}{\partial x^2} \overline{uv} + \frac{\partial^2}{\partial y^2} \overline{uv} + \frac{\partial^2}{\partial z^2} \overline{uv} \right). \quad (39)$$

$$D_{xz}^{2D,3D} = \nu \left(\frac{\partial^2}{\partial x^2} \overline{uw} + \frac{\partial^2}{\partial y^2} \overline{uw} + \frac{\partial^2}{\partial z^2} \overline{uw} \right). \quad (40)$$

$$D_{yz}^{2D,3D} = \nu \left(\frac{\partial^2}{\partial x^2} \overline{vw} + \frac{\partial^2}{\partial y^2} \overline{vw} + \frac{\partial^2}{\partial z^2} \overline{vw} \right). \quad (41)$$

4.4. Components of the turbulent transport tensor

The following terms constitute the various components of the turbulent transport tensor T_{ij} , both in 2D and in 3D:

$$T_{xx}^{2D} = - \left(\frac{\partial}{\partial x} \overline{u^3} + \frac{\partial}{\partial y} \overline{u^2 v} \right), \quad (42)$$

$$T_{xx}^{3D} = - \left(\frac{\partial}{\partial x} \overline{u^3} + \frac{\partial}{\partial y} \overline{u^2 v} + \frac{\partial}{\partial z} \overline{u^2 w} \right). \quad (43)$$

$$T_{yy}^{2D} = - \left(\frac{\partial}{\partial x} \overline{u v^2} + \frac{\partial}{\partial y} \overline{v^3} \right), \quad (44)$$

$$T_{yy}^{3D} = - \left(\frac{\partial}{\partial x} \overline{u v^2} + \frac{\partial}{\partial y} \overline{v^3} + \frac{\partial}{\partial z} \overline{v^2 w} \right). \quad (45)$$

$$T_{zz}^{2D} = - \left(\frac{\partial}{\partial x} \overline{u w^2} + \frac{\partial}{\partial y} \overline{v w^2} \right), \quad (46)$$

$$T_{zz}^{3D} = - \left(\frac{\partial}{\partial x} \overline{u w^2} + \frac{\partial}{\partial y} \overline{v w^2} + \frac{\partial}{\partial z} \overline{w^3} \right). \quad (47)$$

$$T_{xy}^{2D} = - \left(\frac{\partial}{\partial x} \overline{u^2 v} + \frac{\partial}{\partial y} \overline{u v^2} \right), \quad (48)$$

$$T_{xy}^{3D} = - \left(\frac{\partial}{\partial x} \overline{u^2 v} + \frac{\partial}{\partial y} \overline{u v^2} + \frac{\partial}{\partial z} \overline{u v w} \right). \quad (49)$$

$$T_{xz}^{2D} = - \left(\frac{\partial}{\partial x} \overline{u^2 w} + \frac{\partial}{\partial y} \overline{u v w} \right), \quad (50)$$

$$T_{xz}^{3D} = - \left(\frac{\partial}{\partial x} \overline{u^2 w} + \frac{\partial}{\partial y} \overline{u v w} + \frac{\partial}{\partial z} \overline{u w^2} \right). \quad (51)$$

$$T_{yz}^{2D} = - \left(\frac{\partial}{\partial x} \overline{u v w} + \frac{\partial}{\partial y} \overline{v^2 w} \right), \quad (52)$$

$$T_{yz}^{3D} = - \left(\frac{\partial}{\partial x} \overline{u v w} + \frac{\partial}{\partial y} \overline{v^2 w} + \frac{\partial}{\partial z} \overline{v w^2} \right). \quad (53)$$

4.5. Components of the velocity-pressure-gradient tensor

The following terms constitute the various components of the viscous diffusion tensor Π_{ij} , which are the same in 2D and in 3D:

$$\Pi_{xx}^{2D,3D} = - \frac{2}{\rho} \overline{u \frac{\partial p}{\partial x}}. \quad (54)$$

$$\Pi_{yy}^{2D,3D} = - \frac{2}{\rho} \overline{v \frac{\partial p}{\partial y}}. \quad (55)$$

$$\Pi_{zz}^{2D,3D} = - \frac{2}{\rho} \overline{w \frac{\partial p}{\partial z}}. \quad (56)$$

$$\Pi_{xy}^{2D,3D} = - \frac{1}{\rho} \left(\overline{u \frac{\partial p}{\partial y}} + \overline{v \frac{\partial p}{\partial x}} \right). \quad (57)$$

$$\Pi_{xz}^{2D,3D} = - \frac{1}{\rho} \left(\overline{u \frac{\partial p}{\partial z}} + \overline{w \frac{\partial p}{\partial x}} \right). \quad (58)$$

$$\Pi_{yz}^{2D,3D} = - \frac{1}{\rho} \left(\overline{v \frac{\partial p}{\partial z}} + \overline{w \frac{\partial p}{\partial y}} \right). \quad (59)$$

4.6. Components of the pressure strain tensor

The following terms constitute the various components of the pressure strain tensor Π_{ij}^s , which are the same in 2D and in 3D:

$$\Pi_{xx}^{s,2D,3D} = -\frac{2}{\rho} \overline{p \frac{\partial u}{\partial x}}. \quad (60)$$

$$\Pi_{yy}^{s,2D,3D} = -\frac{2}{\rho} \overline{p \frac{\partial v}{\partial y}}. \quad (61)$$

$$\Pi_{zz}^{s,2D,3D} = -\frac{2}{\rho} \overline{p \frac{\partial w}{\partial z}}. \quad (62)$$

$$\Pi_{xy}^{s,2D,3D} = -\frac{1}{\rho} \left(\overline{p \frac{\partial u}{\partial y}} + \overline{p \frac{\partial v}{\partial x}} \right). \quad (63)$$

$$\Pi_{xz}^{s,2D,3D} = -\frac{1}{\rho} \left(\overline{p \frac{\partial u}{\partial z}} + \overline{p \frac{\partial w}{\partial x}} \right). \quad (64)$$

$$\Pi_{yz}^{s,2D,3D} = -\frac{1}{\rho} \left(\overline{p \frac{\partial v}{\partial z}} + \overline{p \frac{\partial w}{\partial y}} \right). \quad (65)$$

4.7. Components of the pressure transport tensor

The following terms constitute the various components of the pressure transport tensor Π_{ij}^t , which are the same in 2D and in 3D:

$$\Pi_{xx}^{t,2D,3D} = -\frac{2}{\rho} \frac{\partial}{\partial x} \overline{p u}. \quad (66)$$

$$\Pi_{yy}^{t,2D,3D} = -\frac{2}{\rho} \frac{\partial}{\partial y} \overline{p v}. \quad (67)$$

$$\Pi_{zz}^{t,2D,3D} = -\frac{2}{\rho} \frac{\partial}{\partial z} \overline{p w}. \quad (68)$$

$$\Pi_{xy}^{t,2D,3D} = -\frac{1}{\rho} \left(\frac{\partial}{\partial y} \overline{p u} + \frac{\partial}{\partial x} \overline{p v} \right). \quad (69)$$

$$\Pi_{xz}^{t,2D,3D} = -\frac{1}{\rho} \left(\frac{\partial}{\partial z} \overline{p u} + \frac{\partial}{\partial x} \overline{p w} \right). \quad (70)$$

$$\Pi_{yz}^{t,2D,3D} = -\frac{1}{\rho} \left(\frac{\partial}{\partial z} \overline{p v} + \frac{\partial}{\partial y} \overline{p w} \right). \quad (71)$$

4.8. Components of the convection tensor

The following terms constitute the various components of the convection tensor C_{ij} , both in 2D and in 3D:

$$C_{xx}^{2D} = U \frac{\partial}{\partial x} \overline{u^2} + V \frac{\partial}{\partial y} \overline{u^2}, \quad (72)$$

$$C_{xx}^{3D} = U \frac{\partial}{\partial x} \overline{u^2} + V \frac{\partial}{\partial y} \overline{u^2} + W \frac{\partial}{\partial z} \overline{u^2}. \quad (73)$$

$$C_{yy}^{2D} = U \frac{\partial}{\partial x} \overline{v^2} + V \frac{\partial}{\partial y} \overline{v^2}, \quad (74)$$

$$C_{yy}^{3D} = U \frac{\partial}{\partial x} \overline{v^2} + V \frac{\partial}{\partial y} \overline{v^2} + W \frac{\partial}{\partial z} \overline{v^2}. \quad (75)$$

$$C_{zz}^{2D} = U \frac{\partial}{\partial x} \overline{w^2} + V \frac{\partial}{\partial y} \overline{w^2}, \quad (76)$$

$$C_{zz}^{3D} = U \frac{\partial}{\partial x} \overline{w^2} + V \frac{\partial}{\partial y} \overline{w^2} + W \frac{\partial}{\partial z} \overline{w^2}. \quad (77)$$

$$C_{xy}^{2D} = U \frac{\partial}{\partial x} \overline{uv} + V \frac{\partial}{\partial y} \overline{uv}, \quad (78)$$

$$C_{xy}^{3D} = U \frac{\partial}{\partial x} \overline{uv} + V \frac{\partial}{\partial y} \overline{uv} + W \frac{\partial}{\partial z} \overline{uv}. \quad (79)$$

$$C_{xz}^{2D} = U \frac{\partial}{\partial x} \overline{uw} + V \frac{\partial}{\partial y} \overline{uw}, \quad (80)$$

$$C_{xz}^{3D} = U \frac{\partial}{\partial x} \overline{uw} + V \frac{\partial}{\partial y} \overline{uw} + W \frac{\partial}{\partial z} \overline{uw}. \quad (81)$$

$$C_{yz}^{2D} = U \frac{\partial}{\partial x} \overline{vw} + V \frac{\partial}{\partial y} \overline{vw}, \quad (82)$$

$$C_{yz}^{3D} = U \frac{\partial}{\partial x} \overline{vw} + V \frac{\partial}{\partial y} \overline{vw} + W \frac{\partial}{\partial z} \overline{vw}. \quad (83)$$

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